Relative Jacobians of Linear Systems

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Matthew Jacob Woolf

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Abstract

Let X be a smooth projective variety. Given any basepoint-free linear system, $|D|$, there is a dense open subset parametrizing smooth divisors, and over that subset, we can consider the relative Picard variety of the universal divisor, which parametrizes pairs of a smooth divisor in the linear system and a line bundle on that divisor. In the case where X is a surface, there is a natural compactification of the relative Picard variety, given by taking the moduli space of pure one-dimensional Gieseker-semistable sheaves with respect to some polarization. In the case of $\mathbb{P}^2$, this is an irreducible projective variety of Picard number 2. We study the nef and effective cones of these moduli spaces, and talk about the relation with variation of Bridgeland stability conditions.

We show how the knowledge of the Picard group of this moduli space of pure one-dimensional sheaves on $\mathbb{P}^2$ can be used to deduce that every section of the relative Picard varieties of the complete linear system of plane curves (of degree at least 3) comes from restriction

a line bundle on $\mathbb{P}^2$. We also give an independent proof of this fact for any basepoint-free linear system on any smooth projective variety when the locus of reducible or non-reduced divisors in the linear system has codimension at least two.

Contents

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1. Introduction

This thesis can be largely divided into two parts. The first part consists of a description of the birational geometry of moduli spaces of pure one-dimensional sheaves on $\mathbb{P}^2$. The second consists of a description of the sections of relative Picard varieties over certain linear systems.

A moduli space of torsion sheaves on $\mathbb{P}^2$ is a space parametrizing pure semistable sheaves with Hilbert polynomial $\chi(\mathcal{F}(m)) = \mu m + \chi$ on $\mathbb{P}^2$ (technically, S-equivalence classes of such sheaves). We will call this space $N(\mu, \chi)$. This space is a compactification of the space of pairs (C, L) with C a smooth plane curve of degree μ and L a line bundle on C with Euler characteristic χ. In the first section of this thesis, we will study the divisor theory of these moduli spaces, specifically, their pseudoeffective and nef cones.

When $\mu < 3$, these spaces have Picard number 1, so these cones are trivial. When $\mu \geq 3$, these spaces have Picard number 2, so their divisor theory is still relatively simple. Moreover, we will show that they are Mori dream spaces, so these spaces have the nicest possible divisor theory.

On these moduli spaces, there is always a natural divisor class which lies on an edge of both the nef and effective cones. Specifically, there is a map from $N(\mu, \chi)$ to the projective space of all curves of degree μ. We can then pull back $\mathcal{O}(1)$ from this projective space to get such a divisor class. Much of this paper is devoted to finding the other edge of both the nef and the pseudoeffective cones. We succeed entirely in calculating the nef cone, but in this thesis we only calculate the effective cone for some choices of μ and χ. In subsequent work with Izzet Coskun and Jack Huizenga, we manage to calculate the effective cones of all moduli spaces of sheaves (not necessarily torsion) on $\mathbb{P}^2$ ([9]).

In doing so, we find something similar to what is known for the Hilbert scheme. That is, the shape of the effective cones is determined by algebraic properties of these sheaves which don't always have an obvious geometric interpretation, while the nef cone is more obviously related to the geometry of these sheaves, thought of as a generalization of line bundles on plane curves.

To be more specific, in the case of the effective cone, we will prove that for many of these moduli spaces, there is a vector bundle E such that the locus of $\mathcal{F} \in N(\mu, \chi)$ such that $h^0(\mathbb{E} \otimes \mathcal{F})$ is a divisor, and it is this divisor which lies on the other edge of the effective cone. The divisor on the other edge of the nef cone is the pullback of an ample divsior by a regular map, and we will describe the fibers of this map in terms of points on curves and the line bundles they define.

For the case of $\chi = 1$, Jinwon Choi and Kiryong Chung have a much more geometric description of the divisor lying on the other edge of the effective cone in [8].

For the calculation of the effective cone, the idea behind the calculation is to use the relationship between line bundles on plane curves, points on plane curves, and points on the plane to show that vector bundles of a certain form have interpolation with respect to the general point of the Hilbert scheme if and only if they have interpolation with respect to the general point of the moduli space of pure one-dimensional sheaves. We use an existence result of Huizenga to show that there are such vector bundles – the locus where interpolation fails is then an effective divisor. To show this divisor is extremal, we construct a moving curve class which has intersection number 0 with this divisor.

The calculation of the nef cone uses the theory of Bridgeland stability conditions. The theory of stability conditions on $\mathbb{P}^2$ gives us a way of writing moduli spaces of sheaves as GIT quotients in a number of ways. Each such way gives us an ample line bundle, so we get a lower bound on the nef cone. We then find a curve which has intersection number 0 with the divisor on the edge of the cone, so this cone must be the entire nef cone.

The spaces $N(\mu, \chi)$ and $N(\mu, \chi')$ are known to be isomorphic if $\chi \cong \pm\chi' \pmod{\mu}$. Intuitively, these isomorphisms come from taking the tensor product of a sheaf with some multiple of $\mathcal{O}(1)$, and from taking a line bundle supported on some plane curve to its dual. If $\mu \neq \mu'$, then the spaces $N(\mu, \chi)$ and $N(\mu', \chi')$ do not have the same dimension, so they cannot be isomorphic. Using the calculation of the nef cone, we will show that if $\chi \ncong \pm\chi' \pmod{\mu}$ and $\mu > 2$, then the dimension of the exceptional locus of a certain canonically defined map distinguishes between $N(\mu, \chi)$ and $N(\mu, \chi')$.

This brings us to the second part of this work, where we discuss an analogue of the Franchetta conjecture for relative Picard varieties of linear systems.

The original Franchetta conjecture was that the only *natural* lines bundles on curves are multiples of the canonical bundle. There are two ways to make this into a precise statement. The weak Franchetta conjecture says that the relative Picard group of the universal curve over M_g, i.e. the group of line bundles on the total space of the universal curve modulo those pulled back from M_g, is generated by the relative canonical bundle. The strong Franchetta conjecture says that the only rational sections of the universal Picard varieties J^d come from multiples of the canonical bundle.

Any line bundle on the universal curve over M_g gives rise to a rational section of the universal Picard variety, so the weak Franchetta conjecture follows from the strong one. However, all known proofs of the strong Franchetta conjecture first use the fact that the weak Franchetta conjecture follows from the calculation of the Picard group of the universal curve by Harer and Arbarello-Cornalba. From the weak Franchetta conjecture, it is possible to deduce that given a rational section of the relative Picard variety, some multiple of it is a multiple of the canonical bundle (see proposition 3.3). The proofs by

Mestrano [21] and Kouvidakis [18] proceed by showing that the degree of a rational section must be a multiple of $2g - 2$ to reduce to the case of J^0, where any section must *a fortiori* be torsion, and then show that the monodromy action on torsion line bundles has no nonzero fixed point.

Given a linear system of curves on a surface, a natural conjecture to make is that the only "natural line bundles" are restrictions of line bundles on the surface (at least when the linear system is basepoint-free). We can also make the same conjecture about linear systems in varieties of dimension greater than two. The purpose of this paper is to show that with one additional necessary hypothesis, this conjecture is correct.

Theorem 1.1. *Let X be a smooth projective variety over $\mathbb{C}$, $|D|$ a basepoint-free linear system of divisors on X such that the locus of divisors which are either reducible or non-reduced has codimension at least two. Then rational sections of the relative Picard variety all come from restricting line bundles on X.*

This theorem can be thought of as a weakening of the Lefschetz hyperplane theorem – there are no hypotheses about ampleness or dimension, but we only get a result about the relative Picard group, not the Picard groups of each divisor. In the case of curves, though, where the Lefschetz hyperplane theorem tells you very little, this result provides the most information. For example, by considering curves in K3 surfaces which generate the Picard group, we see immediately that the degree of any rational section of the universal Picard variety over the moduli space of curves must have degree divisible by $2g-2$, after which we can use the same methods as [21] to deduce the original strong Franchetta conjecture.

The hypothesis on the dimension of the locus of non-integral divisors might seem a little odd at first, but we will show that there are counterexamples to the theorem without this hypothesis. Moreover, in the case of very ample linear systems on a surface, the Castelnuovo-Kronecker theorem (see [7]) implies that this hypothesis is satisfied except in the case where the image of the map from the surface to projective space is either ruled by lines or a projection of the Veronese surface.

Our proof strategy will be to first prove the analogue of the weak Franchetta conjecture, which is very easy in this setting, then to restrict our section to pencils, where we can use Tsen's theorem to show that any rational section must be a linear combination of the restriction of a line bundle on X and the base locus of the pencil. We then show that in fact we can choose a line bundle on X which gives rise to the section.

It is likely that it is possible to reproduce a version of this result using the theory of normal functions, but this argument has the advantage of being purely algebraic and requiring no ampleness hypotheses.

2. Moduli Spaces of Torsion Sheaves

2.1. Preliminaries on the Moduli Space of Torsion Sheaves.

Definition 2.1. A coherent sheaf on $\mathbb{P}^2$ is called pure of dimension 1 if its support is one-dimensional, and the same is true of any nonzero subsheaf.

A pure one-dimensional sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is called semistable if for all nonzero proper subsheaves $\mathcal{G} \subset \mathcal{F}$, we have

$$\frac{\chi(\mathcal{G})}{\mathrm{ch}_1(\mathcal{G})} \leq \frac{\chi(\mathcal{F})}{\mathrm{ch}_1(\mathcal{F})}$$

and it is called stable if moreover this inequality is always strict. This notion of stability will sometimes be called Simpson (semi)stability. (This agrees with the usual definition of Gieseker-Simpson stability applied to pure one-dimensional sheaves.)

In this section, we will recall a number of facts about such sheaves, most of which can be found in le Potier's paper [19], except where otherwise noted.

Given any semistable sheaf $\mathcal{F}$, there is always a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

called the Jordan-Hölder filtration, such that the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are stable. This filtration is not unique, but the subquotients in any such filtration are the same. Two semistable sheaves are called S-equivalent if the stable sheaves appearing as subquotients in their

Jordan-Hölder filtrations are the same. We note that this filtration is only interesting if the sheaf is strictly semistable, i.e. semistable but not stable, so each stable sheaf is the only object in its S-equivalence class.

There is a moduli space, which we will call $N(\mu, \chi)$, parametrizing S-equivalence classes of pure one-dimensional sheaves which are semistable and have Hilbert polynomial $\chi(\mathcal{F}(m)) = \mu m + \chi$. We will call this space the moduli space of one-dimensional sheaves, or the moduli space of torsion sheaves. In a slight abuse of notation, we will write $\mathcal{F} \in N(\mu, \chi)$ to mean that $\mathcal{F}$ is a semistable pure one-dimensional sheaf with Hilbert polynomial $\chi(\mathcal{F}(m)) = \mu M + \chi$. If $c \in K(\mathbb{P}^2)$ is the class of a pure one-dimensional sheaf, then $N(c)$ will be the moduli space of semistable sheaves with class c, which will be isomorphic to some $N(\mu, \chi)$.

The spaces $N(\mu, \chi)$ are irreducible and factorial of dimension μ^2+1. A dimension count then shows that the generic such sheaf is the push forward of a line bundle on a smooth plane curve. The spaces $N(\mu, \chi)$ are smooth away from the locus of strictly semistable sheaves. The space $N(\mu, \chi)$ is isomorphic to $N(\mu, \chi + \mu)$, with the isomorphism being given by the map $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}(1)$. We also know that the space $N(\mu, \chi)$ is isomorphic to $N(\mu, -\chi)$ by the map

$$\mathcal{F} \mapsto \mathscr{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2})$$

as proved in [20].

Corollary 2.2. *If* $\chi \equiv \pm\chi' \pmod{\mu}$, *then* $N(\mu,\chi) \cong N(\chi,\chi')$.

When $\mu \geq 3$, the Picard group of $N(r,\chi)$ is a free abelian group of rank 2. We will now describe generators for this group. There is a morphism (which we will call the Fitting morphism) $N(\mu,\chi) \to \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(\mu)))$ which sends a sheaf to its support. Here, the scheme-theoretic structure on the support is that given by the Fitting ideal of the sheaf. The pullback of $\mathcal{O}(1)$ from $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^2}(\mu)))$ to $N(\mu,\chi)$ will be denoted $\mathcal{L}_0$. This is one generator of the Picard group.

Let $K(\mathbb{P}^2)$ be the Grothendieck group of coherent sheaves. This is a free Abelian group of rank 3. Let c be the class of elements of $N(r,\chi)$ in $K(\mathbb{P}^2)$, Let $a \in K(\mathbb{P}^2)$. Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^2 \times S$ flat over S such the the restriction to each fiber has class c, i.e. a flat family of sheaves of class c on $\mathbb{P}^2$ parametrized by S. Let π_1 (resp. π_2) be the projection from $\mathbb{P}^2 \times S$ to $\mathbb{P}^2$ (resp. S). Then

$$\det\Big(\sum_i (-1)^i R\pi_{2*}(\mathcal{F} \otimes \pi_1^*(a))\Big)$$

is a line bundle on S.

On $K(\mathbb{P}^2)$, there is a quadratic form given by

$$\langle [E],[E'] \rangle \mapsto \sum_i (-1)^i \chi(\mathrm{Tor}^i(E,E')).$$

Given $b \in K(\mathbb{P}^2)$, let $b^\perp$ be the orthogonal complement to b with respect to this quadratic form. If $a \in c^\perp$, then the line bundle above descends to a line bundle on $N(\mu,\chi)$. This determines a group homomorphism $\lambda : c^{\perp} \to \mathrm{Pic}(N(\mu,\chi))$.

Definition 2.3. The line bundle $\lambda(a)$ is called the determinant line bundle determined by a.

As a first case, we note that $\lambda(-h^2) = \lambda_0$.

Let $\delta = \gcd(\mu,\chi)$. Let $\mathcal{L}_1$ be the line bundle on $N(\mu,\chi)$ corresponding to the class

$$a = \frac{1}{\delta}((-\mu) + \chi h) \in K(\mathbb{P}^2)$$

where $h = [\mathcal{O}_H]$. Then $\mathcal{L}_0$ and $\mathcal{L}_1$ generate the Picard group of $N(\mu,\chi)$. It is clear that $\mathcal{L}_0$ gives one edge of both the nef and effective cones, since it is the pullback of an ample line bundle by a nonconstant regular map to a variety of smaller dimension (when $\mu \geq 3$).

2.2. **The Moduli Spaces are Mori Dream Spaces.** In this section, we will prove that the moduli spaces $N(\mu,\chi)$ are Mori dream spaces. The case when the Picard group is isomorphic to $\mathbb{Z}$ is trivial, so for the rest of this section, we will focus on the case $\mu \geq 3$, when the Picard group has rank two. Our first goal will be to understand the canonical class.

Lemma 2.4. *The canonical class K of $N(\mu,\chi)$ is a negative multiple of $\mathcal{L}_0$.*

Proof. This follows from theorem 8.3.3 of [14] and the preceding discussion. In fact, $K = -3\mu\mathcal{L}_0$. Note that the divisor they call $\mathcal{L}_1$ is in fact a multiple of what we call $\mathcal{L}_0$. □

We will now work to understand the singularities of $N(\mu, \chi)$. The first thing to note is that as proved in sections 4.3 and 4.4 of [14], for all m sufficiently large, there is a k such that $N(\mu, \chi)$ is the good quotient of an open subscheme U_m of the scheme parametrizing quotients of $\mathcal{O}(-m)^k$ with Hilbert polynomial $\mu m + \chi$.

Write $\mathcal{H}$ for $\mathcal{O}(-m)^k$ and K for the kernel of some surjective map $\mathcal{H} \to \mathcal{F}$, with $\mathcal{F}$ some fixed semistable sheaf with Hilbert polynomial $\chi(\mathcal{F}(n)) = \mu n + \chi$. By the same discussion in [14], m can be picked larger than the Castelnuovo-Mumford regularity of $\mathcal{F}$ for all semistable $\mathcal{F}$ with this Hilbert polynomial.

Lemma 2.5. *The scheme* U_m *is nonsingular for* m *sufficiently large.*

Proof. By proposition 4.4.4 of [25], the obstruction space to the point of U_m corresponding to the quotient $\mathcal{H} \to \mathcal{F}$ is given by $\mathrm{Ext}^1(K, \mathcal{F})$, so it suffices to show that this space vanishes.

We have the short exact sequence

$$0 \to K \to \mathcal{H} \to \mathcal{F} \to 0$$

Applying $\mathrm{Hom}(\cdot, \mathcal{F})$, we get a long exact sequence of Ext groups, which in particular gives us the exact sequence

$$\mathrm{Ext}^1(\mathcal{H}, \mathcal{F}) \to \mathrm{Ext}^1(K, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$$

so it suffices to show that the two outside groups vanish.

We have

$$\mathrm{Ext}^1(\mathcal{H},\mathcal{F}) \cong H^1(\mathcal{F}(m))^k$$

but since m is larger than the regularity of $\mathcal{F}$, $H^1(\mathcal{F}(m)) = 0$. By Serre duality,

$$\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F},\mathcal{F}(-3))^*$$

. But $\mathcal{F}(-3)$ is a pure 1-dimensional semistable sheaf with the same first Chern class as $\mathcal{F}$ and smaller Euler characteristic, so by considering the image of a putative map $\mathcal{F} \to \mathcal{F}(-3)$, we will be able to contradict the semistability of $\mathcal{F}$ unless this map is 0. □

By [5], a good quotient of a variety with rational singularities (e.g. a nonsingular variety) also has rational singularities. This shows that $N(\mu,\chi)$ has rational (in particular, Cohen-Macaulay) singularities. By [24], a local ring is Gorenstein if it is Cohen-Macaulay, factorial, and a quotient of a regular local UFD. This last property will certainly be satisfied for any local ring occurring here, so we have shown that $N(\mu,\chi)$ has only Gorenstein singularities.

By theorem 11.1 of [16] (noting that a variety with Gorenstein singularities has an invertible dualizing sheaf), rational Gorenstein singularities are canonical, so we have proved the following result.

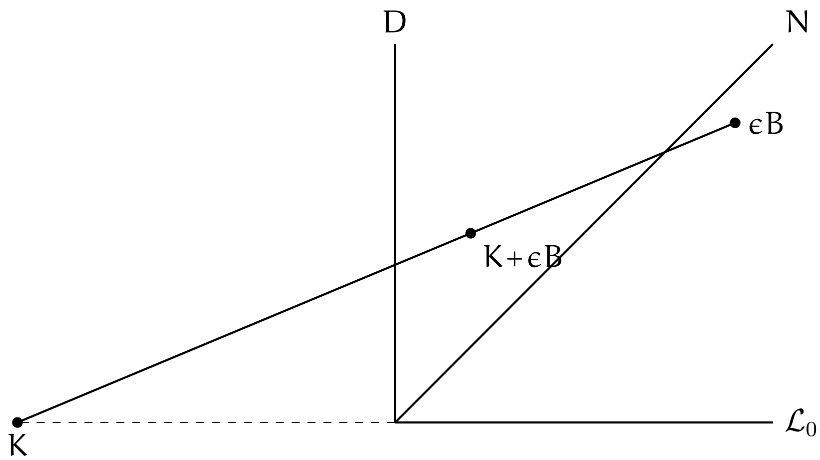
Proposition 2.6. *The space* $N(\mu,\chi)$ *has only canonical (and hence klt, since there is no boundary divisor) singularities.*

By corollary 2.35 of [17], this implies that if D is an effective divisor, and $\epsilon > 0$ is sufficiently small, then $(N(\mu,\chi),\epsilon D)$ is a klt pair.

Definition 2.7. If $D_1, \ldots, D_n$ are Cartier divisors on a variety X, then we will denote by $R(X; D_1, \ldots, D_n)$ the multisection ring of the D_i, i.e.

$$R(X; D_1, \ldots, D_n) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{N}^n} H^0(i_1 D_1 + \cdots i_n D_n)$$

Where there is no ambiguity, we will drop the X from the notation.



Let A be an ample divisor on $N(\mu, \chi)$. Let D be a divisor on the edge of the pseudoeffective cone which does not consist of multiples of $\mathcal{L}_0$. Let N be a divisor on the edge of the nef cone which does not consist of multiples of $\mathcal{L}_0$ By a result of Zariski (lemma 2.8 of [11]), since $\mathcal{L}_0$ and A are both semiample, $R(\mathcal{L}_0, A)$ is finitely generated. Note that any effective divisor which is not a multiple of $\mathcal{L}_0$ can be written as $K + \epsilon B$, with B ample and ϵ arbitrarily small. By choosing B so that ϵ is sufficiently small, we can ensure that $(N(\mu, \chi), \epsilon B)$ a klt pair. By corollary 1.19 of [4], $R(A, D)$ is finitely generated.

Proposition 2.8. *The Cox ring of* $N(\mu, \chi)$ *is finitely generated.*

Proof. We note that the Cox ring of $N(\mu,\chi)$ is isomorphic to the multi-section ring $R(\mathcal{L}_0, D)$. Inside $R(\mathcal{L}_0, D)$, we have two finitely generated subrings, namely $R(\mathcal{L}_0, A)$ and $R(A, D)$. I claim that the generators of these two rings together generate $R(\mathcal{L}_0, D)$. It clearly suffices to show that any section of any effective divisor D on $N(\mu,\chi)$ is in $R(\mathcal{L}_0, A)$ or $R(A, D)$. But D is either in the cone generated by $\mathcal{L}_0$ and A or it is in the cone generated by A and D. In the former case, the section of D lies in $R(\mathcal{L}_0, A)$, and in the latter case, it is contained in $R(A, D)$. $\square$

By [11], we have the following corollary.

Corollary 2.9. *The spaces* $N(\mu,\chi)$ *are Mori dream spaces.*

2.3. **Effective Divisors.** In the case of $N(\mu, 0)$, the line bundle $\mathcal{L}_1 = \lambda(-1)$ is effective – it corresponds to the subvariety of $N(\mu,\chi)$ consisting of sheaves which have a global section ([19]). More generally, we have the following.

Definition 2.10. Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^2$ and E a vector bundle. We say that E is cohomologically orthogonal to $\mathcal{F}$ if $H^i(\mathcal{F} \otimes E) = 0$ for all i. If there is some semistable $\mathcal{F} \in N(\mu,\chi)$ which is cohomologically orthogonal to E, we say that E has the interpolation property for $N(\mu,\chi)$. We will also talk about the interpolation property for the Hilbert scheme of points on $\mathbb{P}^2$, which has a completely analagous definition, since the Hilbert scheme of n points on $\mathbb{P}^2$ is isomorphic to a moduli space of semistable sheaves on $\mathbb{P}^2$ with rank 1, first Chern class 0, and second Chern class $-n$ (see for example [13]).

Proposition 2.11. *Let $c \in K(\mathbb{P}^2)$ denote the class of some semistable pure one-dimensional sheaf $\mathcal{F}$. Suppose E is cohomologically orthogonal to $\mathcal{F}$. Let D be the locus in $N(c)$ of sheaves which are not cohomologically orthogonal to E. Then D is a divisor on $N(\mu, \chi)$, and the corresponding line bundle is the dual of the determinant bundle corresponding to E.*

The proof of this proposition follows from the same argument as in the discussion after proposition 2.10 of [19], which covers the case where $E = \mathcal{O}_{\mathbb{P}^2}$. A calculation with the Hirzebruch-Riemann-Roch formula shows that $\chi(E, \mathcal{F}) = 0$, which is a necessary condition for E to be cohomologically orthogonal to $\mathcal{F}$, if and only if the slope of E is $\frac{-\chi}{\mu}$.

In particular, since the map $\lambda : c^{\perp} \to \mathrm{Pic}(N(\mu, \chi))$ is a homomorphism, the class of D is $a\mathcal{L}_0 + b\mathcal{L}_1$, where a and b are such that

$$\frac{b}{\delta}(-\mu + \chi \operatorname{ch} \mathcal{O}_H) + ah^2 = -\operatorname{ch}(E)$$

More explicitly, we get $a = -\operatorname{ch}_2 E + \frac{\chi}{2\mu} \operatorname{rk} E$ and $b = \frac{\delta}{\mu} \operatorname{rk}(E)$.

Because of the isomorphism $N(\mu, \chi) \cong N(\mu, \chi + \mu)$, in order to calculate the effective cone, it suffices to assume that $0 < \chi \leq \mu$. The isomorphism $N(\mu, \chi) \cong N(\mu, -\chi)$ lets us reduce further to the case $0 \leq \chi \leq \frac{\mu}{2}$. We will find the effective cone in these cases (when we can) by comparing the minimal resolutions of generic one-dimensional sheaves and of generic ideal sheaves of zero-dimensional subschemes. We will use the similarity between the minimal resolutions to show that there is a relation between cohomological orthogonality in the two cases for a particular type of vector bundle.

We now consider the Hilbert scheme $\mathcal{H}_n$ of $n = \binom{\mu+1}{2} + \mu - \chi$ points in $\mathbb{P}^2$.

Theorem 2.12. *Let* $0 \leq 2\chi \leq \mu$. *Let* E *be a vector bundle which fits in a short exact sequence*

$$0 \to k(\mu - \chi)\mathcal{O}(\mu - 2) \to k(2\mu - \chi)\mathcal{O}(\mu - 1) \to E \to 0$$

for some positive integer k. *Such a bundle is called a Steiner bundle. If* E *has the interpolation property with respect to the Hilbert scheme of* n *points in* $\mathbb{P}^2$*, then* $E(-\mu)$ *has the interpolation property with respect to* $N(\mu, \chi)$.

Proof. We note that any length n subscheme Z of $\mathbb{P}^2$ is contained in a curve C of degree μ by an easy parameter count. Suppose that we can pick C smooth, which will be true for the general point of the Hilbert scheme. We get a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-\mu) \to \mathcal{I}_Z \to \mathcal{O}_C(-Z) \to 0$$

where in the last term we consider Z as a Cartier divisor on C. All cohomology groups of E vanish, so we see that $H^i(E \otimes \mathcal{I}_Z) \cong H^i(E \otimes \mathcal{O}_C(-Z))$. The Euler characteristic of $\mathcal{O}_C(-Z)$ is $-\mu^2 + \chi$, so we see that twisting this with $\mathcal{O}_{\mathbb{P}^2}(\mu)$ gives us an element of our desired moduli space.

□

We note that the class of the divisor corresponding to such an E is

$$D = k((\mu - \chi)\mathcal{L}_0 + \delta\mathcal{L}_1)$$

Theorem 7.1 of [12] tells us when a general Steiner bundlehas interpolation with respect to a general ideal sheaf of n points.

Theorem 2.13. *Let $\alpha = 1 - \frac{\chi}{\mu}$. Then there is a bundle E of the above form such that $E(\mu)$ has the interpolation property with respect to the Hilbert scheme of n points in $\mathbb{P}^2$ if and only if either $\alpha > \varphi^{-1}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ or*

$$\alpha \in \left\{\frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \dots\right\}$$

i.e. α is a ratio of consecutive Fibonacci numbers.

2.4. **Moving Curves.** In this previous section, we constructed effective divisors D on $N(\mu, \chi$ for certain choices of μ and χ. In this section, we will show that these divisors are on the edge of the effective cone. For this purpose, we will construct a family of numerically equivalent curves on $N(\mu, \chi)$ which pass through a general of $N(\mu, \chi)$ and have intersection number 0 with D. This will force D to be on the boundary of the effective cone, but since the Picard number of $N(\mu, \chi)$ is two, this means that D will generate an extremal ray of the effective cone. We then need only show that D is not numerically proportional to $\mathcal{L}_0$.

Take a general pencil of plane curves of degree μ. X, the total space of the pencil, is isomorphic to the blowup of $\mathbb{P}^2$ at the μ^2 basepoints of the pencil. Given a line bundle L on X which has Euler characteristic χ when restricted to each fiber, we can push it forward to $\mathbb{P}^2 \times \mathbb{P}^1$ to get a family of semistable sheaves on $\mathbb{P}^2$ parametrized by $\mathbb{P}^1$, which in turn gives us a curve in $N(\mu, \chi)$.

Lemma 2.14. *Suppose that E is a vector bundle with the interpolation property with respect to $N(\mu, \chi)$. Let D be the corresponding line bundle. Let C be as above. Then we have*

$$C \cdot (-D) = \chi(E) + \mathrm{rk}(E)\left(\frac{1}{2}\sum(a_i - a_i^2) + \frac{1}{2}b(b+3)\right) + b\frac{\mathrm{ch}_1(E)}{\mathrm{rk}(E)}$$

Proof. We again use Grothendieck-Riemann-Roch, but now we have

$$\mathrm{ch}(\pi_!(L \otimes E))\,\mathrm{Td}(\mathbb{P}^1) =$$
$$\pi_*[(1 + \sum a_i E_i + bH + \tfrac{1}{2}\left(b^2 - \sum a_i^2\right)H^2)(1 + \tfrac{3}{2}H - \tfrac{1}{2}E + H^2)\,\mathrm{ch}(E)]$$

which equals

$$\chi(E) + \mathrm{rk}(E)\left(\frac{1}{2}\sum(a_i - a_i^2) + \frac{1}{2}b(b+3)\right) + b\frac{\mathrm{ch}_1(E)}{\mathrm{rk}(E)}$$

Again, multiplying by the inverse of $\mathrm{Td}(\mathbb{P}^1)$ does not change this, and since D is given by the dual of the determinant line bundle corresponding to E, we have the desired result. □

Lemma 2.15. *Let $0 < \chi \leq \mu$. Let*

$$L = \sum_{i=1}^{\chi + \frac{1}{2}\mu(\mu-3)} E_i$$

be a line bundle on X*, and* C *the corresponding curve on* $N(\mu,\chi)$*. Then curves in the numerical class of* C *pass through a general point of* $N(\mu,\chi)$*.*

Proof. Let M be a general line bundle of Euler characteristic χ on a general plane curve of degree μ. Since $\chi(M) > 0$, M is linearly equivalent to a sum of (distinct) points $p_1, \ldots, p_n$. We need to show that there is a pencil of curves containing $p_1, \ldots, p_n$, but containing p_i imposes one linear condition on $\mathbb{P}^N$ where

$$N = \frac{1}{2}(\mu+1)(\mu+2) - 1$$

and

$$\frac{1}{2}(\mu+1)(\mu+2) - 1 - \frac{1}{2}\mu(\mu-3) - \chi = 3\mu - \chi$$

will be bigger than 1 for $\chi \leq \mu$. $\square$

Lemma 2.14 lets us work out that the corresponding curves C have intersection number 0 with the divisors associated to the vector bundles of theorem 2.12, since these have Euler characteristic 0 and $a_i = a_i^2$, but by the discussion at the beginning of the section, this means that the divisor D is on the edge of the effective cone.

In the case of $\chi = 0$, we must use a slightly different moving curve. Again, we take a general pencil of plane curves of degree μ, but now we take the line bundle

$$-E_k + \sum_{i=1}^{\frac{1}{2}(\mu-1)(\mu-2)} E_i$$

where E_k is some exceptional divisor which is not included in the second part of the expression. The same arguments as above show that the divisor D gives an edge of the effective cone. We can now prove the following theorem.

Theorem 2.16. *Let* D *be the locus in* $N(\mu, \chi)$ *of sheaves which are not cohomologically orthogonal to a vector bundle coming from theorem 2.12, if such a vector bundle exists. Then* D *and* $\mathcal{L}_0$ *span the two edges of the effective cone of* $N(\mu, \chi)$*. More explicitly, the effective cone is spanned by* $\mathcal{L}_0$ *and*

$$\mathcal{L}_1 + \frac{\chi - \mu}{\delta} \mathcal{L}_0$$

.

Proof. We just need to show that D is not proportional to $\mathcal{L}_0$. But the moving curves we constructed above have intersection number 0 with D and intersection number 1 with $\mathcal{L}_0$, which would be impossible if the two were proportional.

The description of D as a linear combination of $\mathcal{L}_0$ and $\mathcal{L}_1$ comes from the discussion following the statement of theorem 2.12. □

2.5. **Bridgeland Stability Conditions.** In this section, we will give a brief review of the theory of Bridgeland stability conditions, and prove one result about them. I will restrict myself to the case of $\mathbb{P}^2$, and I will often ignore the case of torsion-free sheaves. I will largely follow the treatment of [1] and [6].

For us, $D^b(\mathbb{P}^2)$ will mean the bounded derived category of coherent sheaves on $\mathbb{P}^2$. By $\mathcal{H}^i$, we will mean the ith cohomology sheaf of an object of $D^b(\mathbb{P}^2)$.

Definition 2.17. The heart of a bounded t-structure is a full additive subcategory $\mathcal{A} \subset D^b(\mathbb{P}^2)$ such that for all $A, B \in \mathcal{A}$, $\mathrm{Hom}(A, B[k]) = 0$ if $k < 0$ and for any object $E \in D^b(\mathbb{P}^2)$ there are objects

$$0 = E_m, E_{m+1}, \dots, E_n = E$$

and triangles $E_i \to E_{i+1} \to F_i \to E_i[1]$ such that $F_i[i] \in \mathcal{A}$.

The standard example of the heart of a bounded t-structure on $D^b(\mathbb{P}^2)$ is the full subcategory of coherent sheaves. In general, many of the things which make sense for the category of coherent sheaves work just as well for the heart of any bounded t-structure. For example, a map $A \to B$ with $A, B \in \mathcal{A}$ is an inclusion with respect to $\mathcal{A}$ if the mapping cone is also in $\mathcal{A}$.

Definition 2.18. A stability condition on $\mathbb{P}^2$ is a function $Z : K(\mathbb{P}^2) \to \mathbb{C}$ called a central charge, together with $\mathcal{A}$, the heart of a bounded t-structure on $D^b(\mathbb{P}^2)$, which satisfy the following conditions. First, for any nonzero $A \in \mathcal{A}$, $\mathrm{Arg}(Z(A)) \in \mathbb{R}_{>0}e^{i\theta}$ with $0 < \theta \leq \pi$. We note that this allows us to put a partial order on the arguments which occur, with one argument being bigger than another if it is closer to the negative real axis.

We will call a nonzero $A \in \mathcal{A}$ semistable with respect to the stability condition if every nonzero proper subobject $B \subset A$ has $\mathrm{Arg}(Z(B)) \leq \mathrm{Arg}(Z(A))$. We will call A stable if this inequality is always strict.

The second part of being a stability condition is that any nonzero object $A \in \mathcal{A}$ has a Harder-Narasimhan filtration, i.e. a finite filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = A$$

such that for each of the subquotients $E_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, $\mathrm{Arg}(Z(E_i)) < \mathrm{Arg}(Z(E_{i+1}))$ and the E_i are semistable.

The Harder-Narasimhan filtration of an object of $D^b(\mathbb{P}^2)$ is always unique. Any semistable object has a Jordan-Hölder filtration, i.e. a finite filtration where the subquotients E_i are stable and $\arg(Z(E_i)) = \arg(Z(E_j))$ for all i, j. The Jordan-Hölder filtration is not unique, but the isomorphism classes of the subquotients are well-defined. In general, we will say that two objects are S-equivalent with respect to some stability condition if the isomorphism classes of the subquotients in their Jordan-Hölder filtration are the same up to reordering. We note that if we have an inclusion of two semistable objects $E \subset F$ with $\arg(Z(E)) = \arg(Z(F))$, then E/F if semistable and $\arg(Z(E/F)) = \arg(Z(E))$.

Given a stability condition, we can define a slope function

$$\mu_Z(E) = \frac{-\mathrm{Re}(Z(E))}{\mathrm{Im}(Z(E))}$$

An object $A \in \mathcal{A}$ will then be semistable if and only if for all nonzero proper subobjects $B \subset A$, $\mu_Z(B) \leq \mu_Z(A)$, and it will be stable if and only if this inequality is always strict.

There is another, equivalent, definition of stability condition, which will also prove useful.

Definition 2.19. A slicing of $D^b(\mathbb{P}^2)$ is a collection of full additive subcategories $\mathcal{A}_\phi$ with $\phi \in \mathbb{R}$, the objects of which will be called semistable of phase ϕ, satisfying the following conditions.

(1) For $A \in \mathcal{A}_\phi$, $B \in \mathcal{A}_{\phi'}$ with $\phi' < \phi$, $\mathrm{Hom}(B, A) = 0$.
(2) $\mathcal{A}_\phi[1] = \mathcal{A}_{\phi+1}$
(3) For each nonzero object $C \in D^b(\mathbb{P}^2)$, we have a collection of objects and maps

$$0 = E_0 \to E_1 \to \cdots \to E_n = C$$

such that the mapping cone of $E_i \to E_{i+1}$ is in $\mathcal{A}_{\phi_i}$ with the $\phi_i > \phi_{i+1}$. This collection of maps is called the Harder-Narasimhan filtration of C with respect to the slicing. These mapping cones will be called the subquotients of the filtration.

The following is proposition 5.3 of [6].

Proposition 2.20. *A stability condition on* $D^b(\mathbb{P}^2)$ *is equivalent to a slicing, together with a central charge function* $Z : K(\mathbb{P}^2) \to \mathbb{C}$ *such that for any nonzero* $A \in \mathcal{A}_\phi$, $\arg(Z(A)) = \pi\phi$.

The idea behind the correspondence is as follows. Given a slicing of $D^b(\mathbb{P}^2)$, we can take the full extension-closed subcategory of $D^b(\mathbb{P}^2)$ generated by the objects of $\mathcal{A}_\phi$ with $\phi \in (0,1]$. This will be the heart of a bounded t-structure. To go the other way, for $\phi \in (0,1]$, we let $\mathcal{A}_\phi$ be the full subcategory of objects E in the heart which are semistable and such that $\arg(Z(E)) = \pi\phi$.

Definition 2.21. Given a slicing, for $I \subset \mathbb{R}$ an interval, we can define $\mathcal{A}_I$ to be the extension-closed subcategory of $D^b(\mathbb{P}^2)$ generated by objects in $\mathcal{A}_\phi$ with $\phi \in I$.

A slicing is called locally finite if for all ϕ, there is some $\epsilon > 0$ such that $\mathcal{A}_{(\phi-\epsilon,\phi+\epsilon)}$ is of finite length. A stability condition is called locally finite if the corresponding slicing is locally finite.

In general, we can define a metric on the set of (locally finite) stability conditions as follows.

Definition 2.22. Suppose we have two slicings $\mathcal{P}$ and $\mathcal{P}'$ of $D^b(\mathbb{P}^2)$. Given a nonzero object $E \in D^b(\mathbb{P}^2)$, we will define $\phi^+_{\mathcal{P}}(E)$ (resp. $\phi^-_{\mathcal{P}(E)}$) to be the subquotient of the $\mathcal{P}$-Harder-Narasimhan filtration of E with largest (resp. smallest) phase. We define

$$d(\mathcal{P},\mathcal{P}') = \sup_{0 \neq E \in D^b(\mathbb{P}^2)} \max\left\{|\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{P}'}(E)|, |\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{P}'}(E)|\right\}$$

Given two stability conditions $(Z,\mathcal{P})$ and $(Z',\mathcal{P}')$, we define

$$d((Z,\mathcal{P}),(Z',\mathcal{P}')) = d(\mathcal{P},\mathcal{P}') + \|Z - Z'\|$$

where for the last term of the sum, we use the Euclidean norm on the finite-dimensional vector space $\mathrm{Hom}(K(\mathbb{P}^2), \mathbb{C})$.

Theorem 7.1 of [6] says that the map from the space of (locally finite) stability conditions with the above metric to $\mathrm{Hom}(K(\mathbb{P}^2), \mathbb{C})$ is a local homeomorphism onto a linear subspace.

On $\mathbb{P}^2$, there is a family of stability conditions parametrized by two variables s and t with $s \in \mathbb{R}$ and $t \in \mathbb{R}_{>0}$, which we will think of as giving coordinates on the upper half plane. The central charge is given by

$$Z_{s,t}(E) = -\int_{\mathbb{P}^2} e^{-(s+it)H} \operatorname{ch}(E)$$

The heart of the t-structure is a little more complicated. Given a nonzero torsion-free sheaf E on $\mathbb{P}^2$, we will define its slope $\mu(E)$ to be

$$\frac{c_1(E)H}{\operatorname{rk}(E)}$$

We will say that E is μ-semistable if for all nonzero proper subobjects $F \subset E$, we have $\mu(F) \leq \mu(E)$. This lets us define a notion of Harder-Narasimhan filtration, which we will call the μ-Harder-Narasimhan filtration.

Definition 2.23. For $s \in \mathbb{R}$, define $\mathcal{F}_s$ to be the full subcategory of coherent sheaves A on $\mathbb{P}^2$ which are torsion-free and such that for each subquotient A_i of A in the μ-Harder-Narasimhan filtration of A, $\mu(A_i) \leq s$.

We will define $\mathcal{Q}_s$ to be the full subcategory of coherent sheaves A on $\mathbb{P}^2$ such that for each subquotient A_i of $A/\operatorname{Tor}(A)$ (where $\operatorname{Tor}(A)$ denotes the torsion subsheaf of A) in the μ-Harder-Narasimhan filtration of $A/\operatorname{Tor}(A)$, $\mu(A_i) > s$.

We define a category $\mathcal{A}_s \subset D^b(\mathbb{P}^2)$ as the full subcategory containing objects C with $\mathcal{H}^{-1}(C) \in \mathcal{F}_s$, $\mathcal{H}^0(C) \in \mathcal{Q}_s$, and $\mathcal{H}^i(C) = 0$ for $i \neq 0, 1$.

The following is implied by theorem 5.11 of [1] and the preceding discussion.

Theorem 2.24. *For all $s \in \mathbb{R}$ and $t > 0$, the category $\mathcal{A}_s$ is the heart of a bounded t-structure on $D^b(\mathbb{P}^2)$ and the pair $(Z_{s,t}, \mathcal{A}_s)$ give a stability condition.*

We note that this gives a set-theoretic inclusion from the upper half plane to the space of stability conditions. By inspection, the composite map to $\operatorname{Hom}(K(\mathbb{P}^2), \mathbb{C})$ is a local embedding, so the map from the upper half plane to the space of stability conditions must be continuous.

It will be useful to have a more explicit description of the slope function at each point of the upper half plane. We have

$$\mu_{s,t}(A) = \frac{\frac{\operatorname{rk}(A)}{2}(s^2 - t^2) + \operatorname{ch}_2(A) - s \operatorname{ch}_1(A)}{t(\operatorname{ch}_1(A) - s \operatorname{rk}(A))}$$

Given a sheaf $\mathcal{F}$ (or more generally, an element of $K(\mathbb{P}^2)$), we get a collection of potential walls in the upper half plane. Each potential wall is the locus where $\mu_{s,t}(\mathcal{F}) = \mu_{s,t}(E)$ for some fixed sheaf E. If $\mathcal{F} \in$

$N(\mu, \chi)$, then these potential walls are concentric semicircles. More precisely, we have the following result of [1].

Lemma 2.25. *The potential wall corresponding to a sheaf E is a semicircle with center*

$$\left(\frac{\mathrm{ch}_2(\mathcal{F})}{\mathrm{ch}_1(\mathcal{F})}, 0\right) = \left(\frac{\chi}{\mu} - \frac{3}{2}, 0\right)$$

and radius

$$\sqrt{\left(\frac{\chi}{\mu} - \frac{3}{2}\right)^2 + \frac{2}{\mathrm{ch}_0(E)}\left(\mathrm{ch}_2(E) - \left(\frac{\chi}{\mu} - \frac{3}{2}\right)\mathrm{ch}_1(E)\right)}$$

In particular, the potential walls never intersect.

The following proposition allows us to use the geometry of the potential walls to understand when sheaves have nonzero maps to a torsion sheaf. It extends lemma 6.3 of [1] to the case of torsion sheaves.

Proposition 2.26. *Suppose we have a map $E \to \mathcal{F}$, with $\mathcal{F} \in N(\mu, \chi)$. Suppose at some point (s, t) on the potential wall corresponding to E, $E \to \mathcal{F}$ is an inclusion of semistable objects in the heart of the t-structure. Then this is true for all (s', t') on that potential wall.*

Proof. Let C be the mapping cone of $E \to \mathcal{F}$. We have a distinguished triangle

$$E \to \mathcal{F} \to C \to E[1]$$

Since $\mathcal{F}$ is a sheaf, and both E and C are in $\mathcal{A}_s$, the corresponding long exact sequence of cohomology sheaves shows that E is a sheaf too. By assumption, with respect to (s, t); E, $\mathcal{F}$, and C are all in the same slice, say $\mathcal{A}_\phi$. For pure one-dimensional sheaves and $t > 0$, we have $\phi \in (0, 1)$.

In order to prove the theorem, it is enough to show that for all (s', t') on the wall; E, $\mathcal{F}$, and C will all belong to $\mathcal{A}'_\phi$, that is, the corresponding slice with respect to the slicing given by (s', t'). Since $\phi \in (0, 1]$, this will imply that $E \to \mathcal{F}$ is an inclusion of objects in the heart, and the fact that they're in the corresponding slice implies that they are all semistable of the same slope.

In order to show this, it is enough to show that E, $\mathcal{F}$, and C all belong to $\mathcal{A}'_\phi$ for all stability condition (s', t') on the wall which are contained in a ball of radius $\frac{\epsilon}{2}$ centered at (s, t), where the ball is defined with respect to the above metric on the space of stability conditions and $\epsilon < \min\{phi, 1 - \phi\}$. This is because the metric on the space of stability conditions defines the usual topology on the wall, so any two points (s, t) and (s', t') on the wall can be connected by a sequence of points $(s, t) = (s_0, t_0), (s_1, t_1), \ldots, (s_n, t_n) = (s', t')$ such that $d((s_i, t_i), (s_{i+1}, t_{i+1})) < \frac{\epsilon}{2}$.

Suppose that there are two stability conditions (s, t) and (s', t') within $\frac{\phi}{2}$ of each other such that E, $\mathcal{F}$, and C are all in the same (s, t)-slice but not the same (s', t')-slice. By definition of the potential wall, the argument of the central charge will be the same, so E, $\mathcal{F}$, or C must either be semistable in $\mathcal{A}_{\phi+n}$ for some $n \in 2\mathbb{Z}$, or one of them must stop being semistable. By definition of the metric on the stability manifold, for (s', t') within $\frac{\phi}{2}$ of (s, t), the first case is impossible.

We claim that if $\mathcal{F}$ remains semistable, then so must E and C. Suppose E stops being semistable. Let E' be the semistable factor with the largest slice in the Harder-Narasimhan filtration with respect to the (s',t')-slicing. Then the phase of E' must be larger than ϕ, since the overall phase of E is ϕ. By definition of a slicing, this means that the map $E' \to \mathcal{F}$ is 0. By applying $\mathrm{Hom}(E',\cdot)$ to the above triangle, we see that there must be a nonzero map $E' \to C[-1]$. With respect to (s,t), $C[-1] \in \mathcal{A}_{\phi-1}$, and for (s',t') within $\frac{\epsilon}{2}$ of (s,t), the largest subquotient occurring in the (s',t')-Harder-Narasimhan filtration of $C[-1]$ will have a (s',t')-phase less than $\phi - \frac{1}{2}$, but this means that any map $E' \to C[-1]$ vanishes, a contradiction. A similar argument shows that if $\mathcal{F}$ continues to be semistable, then so must C.

We now want to show that $\mathcal{F}$ is (s',t')-semistable of phase ϕ. Suppose it's not. Let $\mathcal{F}'$ be the piece of largest phase in the (s',t')-Harder-Narasimhan filtration of $\mathcal{F}$. By nestedness of potential walls, the (s,t)-slope of $\mathcal{F}'$ is still greater than the (s,t)-slope of $\mathcal{F}$. Let $\mathcal{G}$ be piece of the (s,t)-Harder-Narasimhan filtration of $\mathcal{F}'$ with greatest (s,t)-phase. By our choice of ϵ, $\mathcal{G}$ will be contained in $(0,1]$. Since the (s,t)-slope of $\mathcal{G}$ is bigger than the (s,t)-slope of $\mathcal{F}'$, we can conclude that the (s,t)-phase of $\mathcal{G}$ is bigger than ϕ.

Since $\mathcal{F}$ is (s,t)-semistable of phase ϕ, we must have the map $\mathcal{G} \to \mathcal{F}$ is zero. Let K be the mapping cone of $\mathcal{F}' \to \mathcal{F}$. Then we have a distinguished triangle

$$K[-1] \to \mathcal{F}' \to \mathcal{F} \to K$$

Applying the functor $\mathrm{Hom}(\mathcal{G}, \cdot)$ to this triangle, and using the fact that $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$, but $\mathrm{Hom}(\mathcal{G}, \mathcal{F}') \neq 0$, we see that there must be a nonzero map $\mathcal{G} \to K[-1]$. We know that the largest piece of the (s', t')-Harder-Narasimhan filtration of K must have phase at most $\phi + \frac{\epsilon}{2}$, so the largest piece of the (s, t)-Harder-Narasimhan filtration of $K[-1]$ can have phase at most $\phi - \frac{1}{2}$, and so there can be no nonzero maps $\mathcal{G} \to K[-1]$, a contradiction. $\square$

2.6. **Nef Divisors.** We recall the following from [2].

Theorem 2.27. *Let Z be a Bridgeland stability condition on $D^b(\mathbb{P}^2)$. Let $v \in K(\mathbb{P}^2)$. Let C be a curve, and $\mathscr{C} \in D^b(C \times \mathbb{P}^2)$. We can associate to this a real number*

$$\Im\left(\frac{-Z(\phi_{\mathscr{C}}(\mathcal{O}_C))}{Z(v)}\right)$$

where $\phi_{\mathscr{C}} : D^b(C) \to D^b(\mathbb{P}^2)$ denotes the integral transform with kernel $\mathscr{C}$.

This assignment induces a map $N_1(\mathcal{M}_Z(v)) \to \mathbb{R}$, or equivalently, gives an element of $N^1(\mathcal{M}_Z(v))$. This element is nef, and it has intersection number 0 with a curve if and only if the curve parametrizes a family of strictly Z-semistable complexes all of which are S-equivalent.

In the specific case of $\mathbb{P}^2$, there is another description of this nef divisor. Using the Beilinson equivalence between $D^b(\mathbb{P}^2)$ and the derived category of representations of a quiver with relations, it is possible to construct moduli spaces of Bridgeland-semistable objects as GIT moduli spaces. Each Bridgeland stability condition gives an ample $\mathbb{R}$-divisor on the relevant GIT quotient. By finding the first GIT wall-crossing, we will have an algebraic contraction of the moduli space of sheaves. Pulling back an ample line bundle on the target will give us the other edge of the nef cone. For a more detailed description of this idea, and of the correspondence between walls in the Bridgeland stability manifold of $\mathbb{P}^2$ and the stable base locus decomposition of the pseudoeffective cone of moduli spaces of sheaves on $\mathbb{P}^2$, see [3], though note that there are some errors.

We already know that $\mathcal{L}_0$ is on the edge of the nef cone, so it remains to find the other edge of the nef cone. To do this, we will find a Bridgeland stability condition (s, t) such that all Simpson-semistable sheaves are (s, t)-semistable, and there is a one-dimensional family of sheaves which are S-equivalent with respect to the (s, t)-stability condition, but not with respect to Simpson stability. The above theorem will then give us a divisor class which is nef but has intersection number 0 with some curve, and hence must lie on the edge of the nef cone. It will then suffice to check that this divisor class is not a multiple of $\mathcal{L}_0$. By theorem 2.8, this divisor class will be a positive real multiple of the class of a semiample line bundle.

Lemma 2.28. *Let $\mathcal{F}$, $\mathcal{G}$ be pure one-dimensional sheaves, and (s,t) a stability condition. If $\mathcal{F} \to \mathcal{G}$ destabilizes $\mathcal{G}$ with respect to (s,t), then it also destabilizes $\mathcal{G}$ with respect to Simpson stability.*

Proof. We have
$$\mu_{s,t}(\mathcal{F}) = \frac{1}{t}\left(\frac{\mathrm{ch}_2(\mathcal{F})}{\mathrm{ch}_1(\mathcal{F}) - s}\right)$$
and similarly for $\mathcal{G}$. But $\mathrm{ch}_2(\mathcal{F}) = \chi(\mathcal{F}) - \frac{3}{2}\mathrm{ch}_1(\mathcal{F})$. This agrees with our earlier definition of Simpson stability. □

The long exact sequence of cohomology sheaves associated to a destabilizing triangle, along with the fact that all the objects in the triangle belong to the heart $\mathcal{A}_s$ together imply that any Bridgeland-destabilizing object must itself be a sheaf. The next lemma shows that for sufficiently large t, torsion-freesheaves cannot be Bridgeland destabilizing. Combined with the previous lemma, this will show that for sufficiently large t, Simpson-stability implies Bridgeland stability, and similarly for semistability.

Lemma 2.29. *Fix the numerical invariants for a pure one-dimensional sheaf. Then the radius of the wall in the stability manifold given by a torsion-free sheaf E is bounded. More specifically, the radius of the wall associated to E is at most*
$$\frac{\mathrm{ch}_1(\mathcal{F})}{2\,\mathrm{rk}(E)}$$

Proof. Let $d = \mathrm{ch}_2(\mathcal{F}) = \chi - \frac{3}{2}\mu$. Suppose we also fixed the Chern character of the destabilizing object E to be (r', c', d'). Then the potential wall is a semicircle with center

$$\left(\frac{d}{mu}, 0\right)$$

and radius

$$\sqrt{\frac{d^2}{\mu^2} + \frac{2d'}{r'} - \frac{2c'd}{r'\mu}}$$

If this is an actual wall, then by proposition 2.26, $E \to \mathcal{F}$ is an inclusion of objects in $\mathcal{A}_s$ at every point of the wall.

Let us denote by R the radius of the wall, and K the kernel of the map $E \to \mathcal{F}$. We have the string of inequalities

$$\frac{d}{\mu} + R < \frac{c_1(E)}{\operatorname{rk}(E)} \le \frac{\mu}{\operatorname{rk}(E)} + \frac{c_1(K)}{\operatorname{rk}(K)} \le \frac{\mu}{\operatorname{rk}(E)} + \frac{d}{\mu} - R$$

which together give us the inequality

$$R < \frac{\mu}{2\operatorname{rk}(E)}$$

□

The following lemma follows from the calculation in [1] of the stability conditions on $\mathbb{P}^2$ for which every ideal sheaf is stable.

Lemma 2.30. *Let $\mathcal{I}_W(k)$ be a twist of an ideal sheaf. Then $\mathcal{I}_W(k)$ is (s,t)-semistable if (s,t) is to the left of the line $s = k$ and (s,t) is outside the potential wall of $\mathcal{O}(k-1)$, which is a semicircle with center*

$$\left(k - |W| - \frac{1}{2}, 0\right)$$

and radius $|W| - \frac{1}{2}$

We will now find the destabilizing object which gives the largest wall for sheaves in $N(\mu, \chi)$. Write $a = d + \frac{1}{2}\mu^2$. If $\mathcal{F}$ is the pushforward of a line bundle of degree a on a smooth plane curve of degree μ, then $\mathcal{F} \in N(\mu, \chi)$. Now write $a = b\mu + \epsilon$ with $-\frac{\mu}{2} \leq \epsilon \leq \frac{\mu}{2}$. Note that the case $\epsilon = \pm\frac{\mu}{2}$ is ambiguous. Since we will be looking at many rank one walls, we note that the potential wall associated to $\mathcal{I}_W(k)$ is a semicircle with center

$$\left(b + \frac{\epsilon}{\mu} - \frac{\mu}{2}, 0\right)$$

and radius

$$\sqrt{\left(\frac{d}{\mu} - k\right)^2 - 2|W|} = \sqrt{\left(\frac{\epsilon}{\mu} - \frac{\mu}{2} + (b - k)\right)^2 - 2|W|}$$

Proposition 2.31. *If* $\epsilon \leq 0$*, then* $\mathcal{I}_W(b)$ *is destabilizing and gives the biggest wall, where* $|W| = |\epsilon|$*. If* $\epsilon \geq 0$*, then* $\mathcal{O}(b)$ *is destabilizing and gives the biggest wall.*

Proof. In order to prove this, there are a number of things we must check. First, we must check that for some one-dimensional sheaf $\mathcal{F}$, this is an inclusion of semistable objects of the same phase. We first consider the case $\epsilon \leq 0$. We want to check that on the potential wall corresponding to $\mathcal{I}_W(b)$, the ideal sheaves are semistable, and there is an inclusion of objects in $\mathcal{A}_s$.

The potential wall associated to $\mathcal{I}_W(b)$ with $|W| = -\epsilon$ has radius $\frac{\epsilon}{\mu} + \frac{\mu}{2}$. We know that the points $\left(b + \frac{2\epsilon}{\mu}, 0\right)$ and $(b - \mu, 0)$ are on the potential wall. These are to the left of the line $s = b$. They strictly contain the semicircle with center $(b - \epsilon - \frac{1}{2}, 0)$ and radius $\epsilon - \frac{1}{2}$ unless $\epsilon = -\frac{\mu}{2}$, in which case the semicircles are equal.

Let C be a smooth plane curve. Consider the line bundle $bH - p_1 \cdots - p_\epsilon$. This gives a semistable sheaf $\mathcal{F}$ of the right numerical invariants, and we have a map $\mathcal{I}_{p_1,\dots,p_\epsilon}(b) \to \mathcal{F}$ which is surjective with kernel $\mathcal{O}(b - \mu)$. Since the left edge of the wall is $(b - \mu, 0)$, we see that when $t > 0$, the mapping cone will be in $\mathcal{A}_s$.

The case where $\epsilon \geq 0$ is very similar, but slightly easier. In this case, the destabilizing object is the line bundle $\mathcal{O}(b)$, which is stable if $s < b$. The potential wall associated to $\mathcal{O}(b)$ has radius $\frac{\mu}{2} - \frac{\epsilon}{\mu}$, so we see that $\mathcal{O}(b)$ will be stable along this potential wall, since the right edge is $(b, 0)$. If we have a line bundle of the form $bH + p_1 + \cdots + p_\epsilon$, then we get a map $\mathcal{O}(b) \to \mathcal{F}$ which has kernel $\mathcal{O}(b - \mu)$ and cokernel a zero-dimensional sheaf. Since the left edge of the potential wall is $(b - \mu + \frac{2\epsilon}{\mu}, 0)$, this will always be in the heart at every point of the potential wall.

We must now check that there can be no bigger destabilizing walls. This will also imply that $\mathcal{I}_W(b)$ really destabilizes the sheaves described in the previous paragraph. The largest possible radius of a higher rank wall is $\frac{\mu}{4}$, but when $\mu > 2$, we have $\frac{\epsilon}{\mu} + \frac{\mu}{2} > \frac{\mu}{4}$, so all higher rank walls are strictly smaller.

Now suppose that $\mathcal{I}_Z(k)$ gives a bigger wall. Since this must be in $\mathcal{A}_s$ for $s = b + \frac{2\epsilon}{\mu}$, we must have $k \geq b + \frac{2\epsilon}{\mu}$. Let K be the kernel of the map $\mathcal{I}_Z(k) \to \mathcal{F}$. We have that K is a torsion-free rank 1 sheaf, and $k - \mu \leq c_1(K) \leq k$. Since the mapping cone of $\mathcal{I}_Z(k) \to \mathcal{F}$ must be in $\mathcal{A}_s$ when $s = b - \mu$, we have that $c_1(K) \leq b - \mu$, and so $k \leq b$. We see that $b = k$ unless $\epsilon = \frac{-\mu}{2}$, in which case b can be $k - 1$. We note that when $\epsilon = -\frac{\mu}{2}$, both $\mathcal{O}(b-1)$ and $\mathcal{I}_W(b)$ with $|W| = \frac{\mu}{2}$ give the same semicircle.

Increasing the number of points in the ideal sheaf decreases the radius of the potential wall, so it suffices to show that we can't have $\mathcal{I}_Z(b)$ with $|Z| < |W|$. In the case of $\epsilon \geq 0$ (and hence also $\epsilon = -\frac{\mu}{2}$), the destabilizing object is a line bundle, so this case is complete. From the inequalities above, we know that $c_1(K) \leq b - \mu$. But if $\epsilon \leq 0$, then $(b - \mu, 0)$ is on the left edge of the wall associated to $\mathcal{I}_W(b)$, so for a bigger wall, the mapping cone will not be in the heart along the leftmost part of the wall.

□

The above proof also gives the following result.

Corollary 2.32. *Suppose $-\frac{\mu}{2} < \frac{\epsilon}{\mu} \leq 0$. Then $\mathcal{I}_W(b)$ with $|W| = |\epsilon|$ are the only destabilizing objects along the largest wall. The Jordan-Hölder factors of $\mathcal{F}$ for sheaves which become semistable along this wall are $\mathcal{I}_W(b)$ and $\mathcal{O}(b-\mu)[1]$. If $0 \leq \frac{\epsilon}{\mu} < \frac{\mu}{2}$, then the only destabilizing objects along the largest wall are of the form $\mathcal{O}(b)$, and the Jordan-Hölder subquotients are $\mathcal{O}(b)$ and an extension of a zero-dimensional sheaf by $\mathcal{O}(b-c)[1]$. If $\frac{\epsilon}{\mu} = \pm\frac{\mu}{2}$, then the destabilizing objects can be can be of either form. In either case,* *the Jordan-Hölder subquotients are again of the same form as above, unless all the points in the ideal sheaf or the zero-dimensional sheaf are collinear, in which case the Jordan-Hölder quotients are* $\mathcal{O}(b-1)$, $\mathcal{O}(b-\mu)[1]$, *and* $\mathcal{O}_\ell(b-\frac{\mu}{2})$, *where* ℓ *is the line containing all the points.*

From the above, we see that the pushforward of a line bundle on a plane curve becomes semistable if the difference with the closest multiple of the hyperplane class is effective. In particular, the general one-dimensional sheaf does not become semistable unless $\mu = 3$, because this is the only case when a general line bundle of degree 1 is effective. In particular, this line bundle defines a birational morphism, so it cannot be a multiple of $\mathcal{L}_0$. In the case of $c = 3$, we see that we get a morphism $N(3,1) \to \mathbb{P}^2$, which for a smooth plane curve remembers the point defining the corresponding line bundle on that curve.

2.7. **Isomorphic Moduli Spaces.** When are two moduli spaces of one-dimensional sheaves $N(\mu,\chi)$ and $N(\mu',\chi')$ isomorphic? The dimension of $N(\mu,\chi)$ is μ^2+1, so a necessary condition is that $\mu = \mu'$. If $\mu = \mu'$ and $\chi \equiv \pm\chi' \pmod{\mu}$, then we have seen that $N(\mu,\chi) \cong N(\mu,\chi')$ in corollary 2.2.

This condition is always satisfied for $\mu = 1$. For $\mu = 2$, we have $N(\mu,0) \cong N(\mu,1) \cong \mathbb{P}^5$, so this condition is not necessary. However, we will show that this is the only case.

Theorem 2.33. *Let* $\mu \geq 3$. *Then* $N(\mu, \chi) \cong N(\mu, \chi')$ *only if* $\chi \equiv \pm\chi' \pmod{\mu}$.

Proof. Let a be as in the previous section, and a' the analogue for χ'. Then $a = \frac{1}{2}\mu(\mu - 3) + \chi$, so $\chi \equiv \pm\chi' \pmod{\mu}$ if and only if $a \equiv \pm a' \pmod{\mu}$. Write $a = b\mu + \epsilon$ with $\frac{-\mu}{2} \leq \epsilon \leq \frac{\mu}{2}$. We note that there is a possible ambiguity if $\epsilon = \frac{\mu}{2}$. Similarly, write $a' = b'\mu + \epsilon'$. It suffices to show that we must have $|\epsilon| = |\epsilon'|$.

Let us suppose that $N(\mu, \chi) \cong N(\mu, \chi')$. This isomorphism induces an isomorphism of Picard groups which must preserve the nef cone. Since one edge of the nef cone is distinguished by the fact that it is proportional to the canonical class, this isomorphism must preserve the other edge of the nef cone too.

This implies that both the map s which sends a sheaf to its support and the map f given by the other edge of the nef cone are preserved by isomorphisms, as is E, be the exceptional locus of f. Let us consider the restriction of s to E. By the discussion in the above section, the general fiber has dimension $|\epsilon|$, since we can identify the fiber over a point in $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(\mu))$ corresponding to a smooth plane curve C with the image of the $|\epsilon|$-th symmetric power of C in its Jacobian under the Abel-Jacobi map. This shows that $|\epsilon|$ must equal $|\epsilon'|$ □

Note that a section of the Fitting morphism would give a (birational) isomorphism between moduli spaces of torsion sheaves. It is possible to use this observation, together with ideas of Mestrano and

Kouvidakis to recover the result that the only sections of the relative Picard varieties of the universal plane curve of degree $d \geq 3$ come from sending C to $\mathcal{O}_{\mathbb{P}^2}(m)$ for some m. In the sequel, we will use a different approach to prove a version of this result in a much more general setting.

3. Franchetta for Linear Systems

3.1. **Relative Picard Varieties.** For the basic facts about relative Picard varieties we recall in this section, we refer the reader to [15] except where otherwise noted. For an introduction to Brauer groups, see for example [23].

Given a smooth projective morphism of varieties $\pi : \mathscr{D} \to S$, we can define the relative Picard variety, which is a countable disjoint union of projective varieties $\mathrm{Pic}(\mathscr{D}/S)^{\eta}$. The components of the relative Picard variety form a group, which we will call the relative Neron-Severi group, $\mathrm{NS}(\mathscr{D}/S)$.

Proposition 3.1. *Let* $\pi : \mathscr{D} \to S$ *be a smooth projective morphism. Let* $L \in \mathrm{Pic}(\mathscr{D})$ *restrict to the trivial line bundle on each fiber of* π*. Then* $L \cong \pi^* L'$ *for some* $L' \in \mathrm{Pic}(S)$*.*

Proof. Consider $L' = \pi_* L$. By Grauert's theorem (corollary (III, 12.9) of [10]) , this is a line bundle, since there is one section of the trivial bundle on an integral scheme. Now consider

$$\pi^* L' = \pi^* \pi_* L \to L$$

This is a nonzero map of invertible sheaves, and it is an isomorphism on fibers by Grauert's theorem, so it is an isomorphism of sheaves.

□

The following is corollary 1.5 of [22] in the case where the fibers are curves, but the proof is the same in general.

Proposition 3.2. *Let* $\pi : \mathscr{D} \to S$ *be a smooth projective morphism. Let* $\tau \in \mathrm{NS}(\mathscr{D}/S)$*. Let* $\sigma : S \to \mathrm{Pic}^{\tau}(\mathscr{D}/S)$ *be a rational section of the natural map* $\mathrm{Pic}^{\tau}(\mathscr{D}/S) \to S$*. Then* σ *extends to a regular section.*

Proposition 3.3. *Let* $\pi : \mathscr{D} \to S$ *be a smooth projective morphism with* S *a smooth base. Let* $\tau \in \mathrm{NS}(\mathscr{D}/S)$*. Let* $\sigma : S \to \mathrm{Pic}^{\tau}(\mathscr{D}/S)$ *be a section. There is a natural number* m *such that* $\sigma^{\otimes m}$ *comes from a line bundle on* $\mathscr{D}$*.*

Proof. The obstruction to σ coming from a line bundle on $\mathscr{D}$ is an element of the Brauer group of S, and taking the tensor product of two sections adds these obstructions, but every element of the Brauer group of S is torsion. □

Proposition 3.4. *Let* $\mathscr{D} \to S$ *a family of smooth projective varieties, with* S *a smooth curve. Let* $\tau \in \mathrm{NS}(\mathscr{D}/S)$*. Let* $\sigma : S \to \mathrm{Pic}^{\tau}(\mathscr{D}/S)$ *be a section. Then there is a line bundle* L *on* $\mathscr{D}$ *which gives rise to* σ*.*

Proof. The obstruction to σ coming from a line bundle on $\mathscr{D}$ is an element of the Brauer group of S, but Tsen's theorem says that the Brauer group of a curve over an algebraically closed field is trivial.

□

Suppose we have a family of projective varieties $\pi : \mathscr{D}/S$ over an integral base such that the general fiber is smooth. By abuse of notation, we will refer to the relative Picard variety (resp. Neron-Severi group) of the restriction of π to the complement of the discriminant locus in S as the relative Picard variety(resp. Neron-Severi group) of π.

3.2. **Counterexamples.** In this section, we show that the hypothesis on the dimension of the locus of non-integral divisors in theorem 1.1 is necessary.

The first counterexample is very simple. Take the complete linear system of conics in $\mathbb{P}^2$. There is certainly a rational section of the relative Picard variety which assigns to a smooth conic C the line bundle $\mathcal{O}_C(1)$, but there is no line bundle on $\mathbb{P}^2$ which restricts to $\mathcal{O}_C(1)$ on each conic.

Conics are somewhat exceptional, having genus 0, so we will rest easier once we have found a counterexample using curves of higher genus. Indeed, we will show that there are counterexamples with arbitrarily high genus.

Let S be a very general double cover of $\mathbb{P}^2$ branched over a sextic curve. Then S is a K3 surface of Picard number 1, generated by the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Let C be the preimage of a conic in $\mathbb{P}^2$. Then C is a hyperelliptic curve of genus 5, so $|C| \cong \mathbb{P}^5$, since on a K3 surface, the dimension of a linear system of curves is equal to the genus. Since the linear system of conics in $\mathbb{P}^2$ is five-dimensional, this means that every curve in the linear system is a double cover of a conic.

There is a rational section of $J^2 \to |C|$ which sends each curve to the line bundle on that curve giving rise to the double cover of $\mathbb{P}^1$. On the other hand, it is easy to check that there is no line bundle on S which has intersection number 4 with C, so this rational section cannot come from a line bundle on S.

By the Noether-Lefschetz theorem for weighted projective spaces, a double cover of $\mathbb{P}^2$ branched along a very general curve of degree at least 6 has Picard number 1. Taking the preimage of a conic in such a surface will give a hyperelliptic curve. It is not difficult to show that all curves in the same linear system will again be double covers of conics. The same argument as above shows that the corresponding rational section of $J^2 \to |C|$ cannot come from a line bundle on the surface. Increasing the degree of the branch curve increases the genus of the curves in the linear system, so this provides us with counterexamples of arbitrarily high genus.

3.3. **Proof of Franchetta for Linear Systems.** Let X be a smooth projective variety, and $|D|$ a linear system. The universal divisor $\mathscr{D}$ over $|D|$ maps to X, so we can pull back any line bundle L on X to the universal divisor. By the universal property of the relative Picard scheme, L gives rise to a rational section of the relative Picard variety of line bundles, and its image is contained in some component $\mathrm{Pic}^{\tau}(\mathscr{D}/|D|)$ with $\tau \in \mathrm{NS}(\mathscr{D}/|D|)$.

We will let $|D|^s$ be the complement of the discriminant locus, and $\mathscr{D}^s$ its preimage in $\mathscr{D}$. By Bertini's theorem, if $|D|$ is basepoint-free, then $|D|^s$ is nonempty. For the rest of this section, we will assume that the hypotheses of theorem 1.1 hold.

We first note that the analogue of the weak Franchetta conjecture for basepoint-free linear systems is very easy.

Lemma 3.5. *The relative Picard group (i.e. the Picard group of the total space modulo the Picard group of the base) of the universal divisor over a basepoint-free linear system on a projective variety X is generated by $\mathrm{Pic}(X)$.*

Proof. Since $|D|$ is basepoint-free, the natural map $\mathscr{D} \to X$ realizes $\mathscr{D}$ as a projective bundle over X, so its Picard group is the direct sum of $\mathrm{Pic}(X)$ and the tautological quotient line bundle $\mathcal{O}(1)$, but $\mathcal{O}(1)$ is pulled back from $|D|$. □

By proposition 3.3, this means that there is an integer m such that $\sigma^{\otimes m}$ comes from some line bundle on X, which we will call L.

Proposition 3.6. *For every $[D'] \in |D|^s$, we have $\sigma([D']) = [L_{D'}|_{D'}]$ for some $L_{D'} \in \mathrm{Pic}(S)$.*

Proof. Fix $D' \in |D|$ and consider a general pencil containing D'. Let $\tilde{X}$ be the total space of the pencil, and $\mathbb{P}^1$ the base. $\tilde{X}$ is the blowup of X at the scheme-theoretic base locus of the pencil, which is smooth by a double application of Bertini's theorem, so

$$\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}E_i$$

where the E_i are the connected components of the exceptional locus. Let D_p denote the fiber over a point $p \in \mathbb{P}^1$. The class of D_p in $\mathrm{Pic}(\tilde{X})$ will be $D - E$, where $E = \sum E_i$ is the (reduced) exceptional divisor.

Let $C \subset \mathbb{P}^1$ be the complement of the discriminant locus (or the complement of a point if the discriminant locus is empty). By proposition 3.2, σ is defined on all of C. Let D_C be the preimage of C in $\tilde{X}$. All the fibers of the map $\tilde{X} \to \mathbb{P}^1$ are integral by hypothesis, so

$$\mathrm{Pic}(D_C) \cong \mathrm{Pic}(\tilde{X})/(D - E) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}E_i/(D - E)$$

by the exact sequence of divisor class groups for an open subset ((II, 6.5) of [10]).

By proposition 3.4, σ comes from a line bundle on D_C, $\tilde{L}$. We can write

$$\tilde{L} \equiv L' + \sum a_i E_i \pmod{D - E}$$

with $L' \in \mathrm{Pic}(X)$ by the calculation of $\mathrm{Pic}(D_C)$. We know that

$$mL' + m \sum a_i E_i - L$$

restricts to the trivial line bundle on the fibers of D_C/C, so by proposition 3.1, it is trivial in $\mathrm{Pic}(D_C)$ (since the class of a fiber is trivial), and hence a multiple of $D - E$ in $\mathrm{Pic}(\tilde{X})$ (say $n(D - E)$). We can rewrite this fact as the equation

$$mL' - L + nD + \sum (ma_i - n)E_i = 0$$

in $\mathrm{Pic}(\tilde{X})$. Since all the E_i are linearly independent from each other and from $\mathrm{Pic}(X)$ in $\mathrm{Pic}(\tilde{X})$, this means that in particular all the a_i are equal, say to a. But

$$L' + a \sum E_i = L' + aE$$

has the same restriction to each fiber of D_C/C as $L' + aD$, which is the pullback of a line bundle on X. $\square$

Consider the set

$$T = \{\tau' \in \mathrm{NS}(X) : \tau'|_{|D|} = \tau\}$$

For each $\tau' \in T$, we have a restriction map

$$\mathrm{Pic}^{\tau'}(X) \times |D|^s \to \mathrm{Pic}^{\tau}(\mathscr{D}^s/|D|^s)$$

Each of these maps is proper over $|D|^s$ since

$$\mathrm{Pic}^{\tau'}(X) \times |D|^s \to \mathrm{Pic}^{\tau}(\mathscr{D}^s/|D|^s)$$

is proper over $|D|^s$ (since $\mathrm{Pic}^{\tau'}(X)$ is proper) and the map

$$\mathrm{Pic}^{\tau}(\mathscr{D}^s/|D|^s) \to |D|^s$$

is proper. In particular, each of the restriction maps has a closed image. By the above proposition, the image of σ is contained in the union of these images. But T is a countable set by Severi's theorem of the base, so by pulling back by σ the images of the restriction maps for each $\tau' \in T$, we see that $|D|^s$ is a countable union of closed subvarieties, so one of them must be all of $|D|^s$, and hence there must be some τ' such that the image of σ is contained in the image of $\mathrm{Pic}^{\tau'}(X)$.

Pick $\overline{L} \in \mathrm{Pic}^{\tau'}(X)$ with τ' chosen as above, and let $\sigma_{\overline{L}}$ be the corresponding section of $\mathrm{Pic}^{\tau}(\mathscr{D}^s/|D|^s)$. By considering $\sigma - \sigma_{\overline{L}}$, we might as well assume that τ and τ' are both 0.

Now consider the map

$$r : \mathrm{Pic}^0(X) \times |D|^s \to \mathrm{Pic}^0(\mathscr{D}^s/|D|^s)$$

This is a morphism of abelian varieties over $|D|^s$ which preserves 0, so in particular, it's a group homomorphism. Let K be the kernel of r. Let

$$\pi : \mathrm{Pic}^0(X) \times |D|^s \to |D|^s$$

be the projection onto the second factor. We will need the following lemma.

Lemma 3.7. *There is a nonempty open set $U \subset |D|^s$ such that $K \cap \pi^{-1}(x)$ is constant for $x \in U$.*

Proof. By generic flatness, there is an open set $V \subset |D|^s$ such that

$$K_x = K \cap \pi^{-1}(x) \subset \mathrm{Pic}^0(X)$$

is a flat family of closed subvarieties. We will now restrict our attention to V.

Consider the component of the Hilbert scheme of closed subvarieties of $\mathrm{Pic}^0(X)$ which contains K_x. The tangent space to this point of the Hilbert scheme is given by $H^0(N_{K_x/\mathrm{Pic}^0(X)})$, but since K_x is a closed subgroup, this normal bundle is a trivial bundle of rank equal to c, the codimension of K_x in $\mathrm{Pic}^0(X)$. Therefore, $h^0(N_{K_x/\mathrm{Pic}^0(X)}) = cn$, where n is the number of components of K_x.

We will now construct a flat family of embedded deformations of K_x in $\mathrm{Pic}^0(X)$ such that its base dominates this component of the Hilbert scheme. Assume first that K_x is connected. We note that we can identify the vector space $N_{K_x/\mathrm{Pic}^0(X),0}$ with $T_0(\mathrm{Pic}^0(X)/K_x)$. Let

$$K_x' \subset \mathrm{Pic}^0(X) \times \mathrm{Pic}^0(X)$$

be such that

$$\pi_2^{-1}(\{a\}) = K_x + a$$

i.e. K_x translated by a. This is just the universal family of translates of K_x. There is an induced map from $\mathrm{Pic}^0(X)$, considered as the base of this family, to the Hilbert scheme of subschemes of $\mathrm{Pic}^0(X)$, and the differential of this map at 0 is given by the natural map

$$T_0\,\mathrm{Pic}^0(X) \to T_0(\mathrm{Pic}^0(X)/K_x) \cong H^0(N_{K_x/\mathrm{Pic}^0(X)})$$

which is certainly surjective. Moreover, the kernel of this map consists of directions in which $a \in K_x$, or equivalently, directions in which

$0 \in K_x + a$. In particular, any point near to K_x but not equal to it, cannot be a subgroup, since it will not contain 0.

If K_x is not connected, then nearby points of the Hilbert scheme will correspond to independent translations of each component of K_x, so again, none of the nontrivial deformations of K_x can be a subgroup.

$\square$

Let K_0 be K_x for some x in the U of the above lemma. We get a birational factorization

$$\mathrm{Pic}^0(X) \times |D|^s \to \mathrm{Pic}^0(X)/K_0 \times |D|^s \dashrightarrow \mathrm{Pic}^0(\mathscr{D}^s/|D|^s)$$

where the last arrow is a rational map which is birational onto its image. We know that σ is contained in the closure of the image of this last map, and σ is defined for all points of $|D|^s$ by proposition 3.2, so we see that σ factors birationally to give a map

$$|D|^S \dashrightarrow \mathrm{Pic}^0(X)/K_0.$$

Since $|D|^s$ is an open subvariety of projective space and $\mathrm{Pic}^0(X)/K_0$ is an abelian variety, this map must be constant. We can therefore find an element L' of $\mathrm{Pic}^0(X)$ such that σ and $\sigma_{L'}$ agree on a dense open subset of $|D|^s$, and hence agree everywhere.

References

[1] D. Arcara, A. Bertram, I. Coskun, and J. Huizenga. The Minimal Model Program for the Hilbert Scheme of Points on Pˆ2 and Bridgeland Stability. *ArXiv e-prints*, Mar. 2012.

[2] A. Bayer and E. Macri. Projectivity and Birational Geometry of Bridgeland moduli spaces. *ArXiv e-prints*, Mar. 2012.

[3] A. Bertram, C. Martinez, and J. Wang. The birational geometry of moduli space of sheaves on the projective plane. *ArXiv e-prints*, Jan. 2013.

[4] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[5] J.-F. Boutot. Singularités rationnelles et quotients par les groupes réductifs. *Invent. Math.*, 88(1):65–68, 1987.

[6] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[7] G. Castelnuovo. Sulle superficie algebriche che ammettono un sistema doppiamente infinito di sezioni piane riducibili. *Rend. R. Acc. dei Lincei*, V(3), 1894.

[8] J. Choi and K. Chung. Birational geometry of the moduli space of sheaves, 2013. Personal Correspondence.

[9] I. Coskun, J. Huizenga, and M. Woolf. The effective cone of the moduli space of sheaves on the plane. *ArXiv e-prints*, Jan. 2014.

[10] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[11] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion

of his 60th birthday.

[12] J. Huizenga. Restrictions of Steiner bundles and divisors on the Hilbert scheme of points in the plane. *ArXiv e-prints*, Feb. 2011.

[13] J. Huizenga. Effective divisors on the Hilbert scheme of points in the plane and interpolation for stable bundles. *ArXiv e-prints*, Oct. 2012.

[14] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[15] S. L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[16] J. Kollár. Singularities of Pairs. In *eprint arXiv:alg-geom/9601026*, page 1026, Jan. 1996.

[17] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[18] A. Kouvidakis. The Picard group of the universal Picard varieties over the moduli space of curves. *J. Differential Geom.*, 34(3):839–850, 1991.

[19] J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. *Rev. Roumaine Math. Pures Appl.*, 38(7-8):635–678, 1993.

[20] M. Maican. A Duality Result for Moduli Spaces of Semistable Sheaves Supported on Projective Curves. *ArXiv e-prints*, Dec. 2007.

[21] N. Mestrano. Conjecture de Franchetta forte. *Invent. Math.*, 87(2):365–376, 1987.

[22] N. Mestrano and S. Ramanan. Poincaré bundles for families of curves. *J. Reine Angew. Math.*, 362:169–178, 1985.

[23] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[24] M. P. Murthy. A note on factorial rings. *Arch. Math. (Basel)*, 15:418–420, 1964.

[25] E. Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.